Critical Hysteresis in Random Field XY and Heisenberg Models

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We study zero-temperature hysteresis in random-field XY and Heisenberg models in the zero-frequency limit of a cyclic driving field. We consider three distributions of the random field and present exact solutions in the mean field limit. The results show a strong effect of the form of disorder on critical hysteresis as well as the shape of hysteresis loops. A discrepancy with an earlier study based on the renormalization group is resolved.

I. INTRODUCTION

Hysteresis is common in systems subjected to a cyclic force [1]. It means that the response to a changing force depends on the history of the force. In particular, the response in increasing force is different from that in decreasing force. This is caused by the delay in responding to the force. Theoretically hysteresis should disappear if the force changes sufficiently slowly but this often corresponds to unrealistically long time periods. Several complex and disordered systems such as permanent magnets show hysteresis over the longest practical time scales. A microscopic model of this phenomenon reveals critical points on the hysteresis loop where non-equilibrium susceptibility of the system diverges [2, 3, 5]. These points are characterized by a diverging correlation length, scaling of various quantities, and universality of critical exponents that is reminiscent of equilibrium critical phenomena. Universal behavior covers a wide class of materials, but aspects of universality are best examined in the framework of specific models. We focus on the zero temperature hysteresis in classical spin models in a quenched random field in the limit of zero frequency of the driving field. These restrictions are not so drastic as may appear at first sight. In a pioneering work, Sethna et al [2] used these simplifying features to examine magnetic hysteresis in the random field Ising model. Their model reproduces several experimentally observed features including the shape of hysteresis loops, Barkhausen noise, and return point memory [2–4]. They employ a Gaussian distribution of the random field with mean value zero and standard deviation σ that plays the role of a tuning parameter in the model. They find a critical value σ_c such that for $\sigma < \sigma_c$, each half of the hysteresis loop has a first order jump in the magnetization at some applied field h. The size of the jump goes to zero as $\sigma \to \sigma_c$ from below. If h_c is the critical field at which the jump vanishes, $\{h_c, \sigma_c\}$ is a non-equilibrium critical point showing scaling of thermodynamic functions and universality of critical exponents in its vicinity. There appears to be a fair amount of experimental support for the predictions of this model [3] . Silveira and Kardar [6] generalized this model to n-component continuous spins. Using the renormalization group approach they showed that the generalized model also shows critical hysteresis, and the exponents are independent of n above 6 dimensions. The significance of 6 dimensions is that it is the upper critical dimension of the model with a Gaussian random field [7]. In 6 and higher dimensions the action is adequately described by a quadratic term, and higher order terms are irrelevant in the renormalization-group sense. The quadratic action can be solved exactly and predicts a second order transition as $\sigma \to \sigma_c$ from below. The quadratic action is n-dependent but the critical exponents are not.

Recently we solved [8] a slightly different variant of the n-component spin model exactly in the mean field approximation based on infinitely weak but infinitely long range interactions. Our variant used randomly oriented but fixed length random fields and spins. We were surprised to find strikingly different results from those of Silveira and Kardar [6] for their quadratic action. For $n = 2$ (XY model), we found a first order transition and no critical point. We also found wasp-waisted hysteresis loops in this case. For $n = 3$ (Heisenberg model) as well, we did not find a critical point of the familiar type but one with rather peculiar criticality. The question arises why the predictions of two qualitatively similar models are so different. Is it because of our variant of the random field or due to the use of soft (variable length) n-component spins in the field theoretic formulation of the renormalization group. We are now in a position to resolve this question. We recover the predictions of the quadratic action of Silveira and Kardar if we keep the length of vector spins fixed but allow the magnitude of the random field to have a Gaussian distribution. We also consider a rectangular distribution of the magnitudes of random fields to examine if a distribution with a

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compact support gives a different behavior than an unbounded distribution. Rectangular and Gaussian distributions are known to give qualitatively different results in the case of Ising spins [5, 9]. However, for XY and Heisenberg spins we find these two distributions yield similar results.

II. EQUATIONS OF MOTION

The model studied in [8] is characterized by the Hamiltonian,

$$H = -J \sum_{i,j} \vec{S}_i.\vec{S}_j - \sum_i \vec{h}_i.\vec{S}_i - \vec{h}.\sum_i \vec{S}_i \tag{1}$$

Here $\{\vec{S}_i\}$ are spins located at sites $\{i = 1, 2, \ldots N\}$ of a d-dimensional lattice, $\{\vec{h}_i\}$ are quenched random fields, and $\vec{h}$ is a uniform applied field; $\vec{S}_i$, $\vec{h}_i$, and $\vec{h}$ are n-component vectors with magnitudes $|\vec{S}_i| = 1$, $|\vec{h}_i| = a_i$, and $|\vec{h}| = h$ respectively. The vectors $\{\vec{h}_i\}$ are randomly oriented but their magnitudes $\{a_i\}$ may have one of several different distributions. We consider three cases: (i) a_i is constant independent of the sites ($a_i = a$), (ii) a_i has a Gaussian distribution with standard deviation σ centered at the origin, and (iii) a_i has a uniform distribution in the interval $[0 \leq a_i \leq \Delta]$, and zero otherwise. The discrete time dynamics of fixed-length spins at zero temperature is given by,

$$\vec{S}_i(t+1) = \frac{\vec{f}_i(t)}{|\vec{f}_i(t)|} \qquad \vec{f}_i(t) = J \sum_j \vec{S}_j(t) + \vec{h}_i + \vec{h} \tag{2}$$

Here $\vec{f}_i(t)$ is the effective field at site-i at time t and the factor $|\vec{f}_i(t)|$ in the denominator ensures that the dynamics does not alter the length of the spin vectors. We characterize the system by $m(t) = \frac{1}{N}\sum_i \vec{h}.\vec{S}_i(t)$, the induced magnetization per spin along $\vec{h}$. In the mean field limit a spin S_i interacts with every other spin S_j with strength $J = J_0/N$. Let the applied field $\vec{h}$ be along the x-axis. Then the equations for the evolution of magnetization in XY and Heisenberg model in the three cases of the random field distribution mentioned above are [10]:

A. Randomly oriented fields of fixed magnitude $a_i = a$.

If the quenched fields $\{\vec{h}_i\}$ are randomly oriented vectors of a fixed length a, the equations relating $m(t+1)$ to $m(t)$ are,

$$m(t+1) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\{J_0 m(t) + h\} + a\cos\alpha_i}{[a^2 + 2\{J_0 m(t) + h\} a\cos\alpha_i + \{J_0 m(t) + h\}^2]^{\frac{1}{2}}} d\alpha_i \quad \text{[XY model: } n = 2] \tag{3}$$

$$m(t+1) = \frac{1}{2} \int_0^{\pi} \frac{\{J_0 m(t) + h\} + a\cos\alpha_i}{[a^2 + 2\{J_0 m(t) + h\} a\cos\alpha_i + \{J_0 m(t) + h\}^2]^{\frac{1}{2}}} \sin\alpha_i d\alpha_i \quad \text{[Heisenberg model: } n = 3] \tag{4}$$

Equation (3) is a slight generalization of equation (9) in reference [8] for $a = 1$. One may understand it intuitively as follows. The zero temperature iterative dynamics aligns $\vec{S}_i(t+1)$ along the effective field $\vec{f}_i(t)$ at site-i. The effective field $\vec{f}_i(t)$ is the sum of the mean field $\{J_0 m(t) + h\}$ along the x-axis (the direction of the applied field) and a random field of magnitude a making an angle α_i with the x-axis. Thus the components of $\vec{f}_i(t)$ along the x and y axes are $f_{ix} = \{J_0 m(t) + h\} + a\cos\alpha_i$ and $f_{iy} = a\sin\alpha_i$. The component of the unit vector $\vec{S}_i(t+1)$ along the x-axis contributes to the magnetization $m(t+1)$. Its contribution is equal to the cosine of the angle that $\vec{f}_i(t)$ makes with the x-axis and therefore equal to $f_{ix}/\sqrt{f_{ix}^2 + f_{iy}^2}$. This explains the integrand in equation (3). The integral over α_i amounts to taking an average over all sites to get the magnetization of the system. Equation (4) may be understood similarly. In this case the random fields as well as the spins are three component vectors and we need a polar and an azimuthal angle to specify their orientation. The polar angle α_i of $\vec{h}_i$ is measured from the x-axis which is again the direction of the applied field and the magnetization of the system. The azimuthal angle is integrated out trivially because the component of $\vec{S}_i(t+1)$ along the x-axis does not depend on it leaving us with equation (4).

B. Randomly oriented fields with a Gaussian distribution of a_i

Randomly oriented fields with a Gaussian distribution of a_i means that each Cartesian coordinate of the n-component field $\vec{h}_i$ has a Gaussian distribution with average value zero and standard deviation σ. The integrals in the equation relating $m(t)$ to $m(t+1)$ are performed more conveniently in polar coordinates. We integrate over a_i with appropriate normalization and weight factors for two and three dimensional integrals respectively to get,

$$m(t+1) = \frac{1}{2\pi\sigma^2}\int_0^\infty e^{-\frac{a_i^2}{2\sigma^2}} a_i da_i \int_0^{2\pi} \frac{\{J_0 m(t)+h\} + a_i\cos\alpha_i}{[a_i^2 + 2\{J_0 m(t)+h\}a_i\cos\alpha_i + \{J_0 m(t)+h\}^2]^{\frac{1}{2}}} d\alpha_i \quad \text{[XY: } n=2\text{]} \tag{5}$$

$$m(t+1) = \frac{1}{\sqrt{2\pi}\sigma^3}\int_0^\infty e^{-\frac{a_i^2}{2\sigma^2}} a_i^2 da_i \int_0^{\pi} \frac{\{J_0 m(t)+h\} + a_i\cos\alpha_i}{[a_i^2 + 2\{J_0 m(t)+h\}a_i\cos\alpha_i + \{J_0 m(t)+h\}^2]^{\frac{1}{2}}} \sin\alpha_i d\alpha_i \quad \text{[Heisenberg: } n=3\text{]} \tag{6}$$

C. Randomly oriented fields with a rectangular distribution of a_i.

The equations for this case are obtained on the same lines as in the preceding section by integrating over a_i with a uniform distribution in the interval $[0 \le a_i \le \Delta]$.

$$m(t+1) = \frac{1}{\pi\Delta^2}\int_0^\Delta a_i da_i \int_0^{2\pi} \frac{\{J_0 m(t)+h\} + a_i\cos\alpha_i}{[a_i^2 + 2\{J_0 m(t)+h\}a_i\cos\alpha_i + \{J_0 m(t)+h\}^2]^{\frac{1}{2}}} d\alpha_i \quad \text{[XY: } n=2\text{]} \tag{7}$$

$$m(t+1) = \frac{3}{2\Delta^3}\int_0^\Delta a_i^2 da_i \int_0^{\pi} \frac{\{J_0 m(t)+h\} + a_i\cos\alpha_i}{[a_i^2 + 2\{J_0 m(t)+h\}a_i\cos\alpha_i + \{J_0 m(t)+h\}^2]^{\frac{1}{2}}} \sin\alpha_i d\alpha_i \quad \text{[Heisenberg: } n=3\text{]} \tag{8}$$

III. HYSTERESIS IN XY MODEL

Although the integral in equation (3) can be written in terms of elliptic functions of the first and second kind, it has an appealing geometrical interpretation [8, 11] in its present form. The left-hand-side represents the average projection of a spin along the x-axis. This is proportional to two forces acting along the x-axis (i) mean field $\{J_0 m(t) + h\}$ and (ii) a random field "$a\cos\alpha_i$". These account for the numerator in the integrand. The denominator ensures that the resultant of the mean field and the random field vectors is a unit vector. The orientation of the random field changes from site to site and the integral over α_i represents an average over it. In order to examine the structure of a fixed point m^* of equation (3), it is convenient to set $J_0 = 1$ and $h = 0$ so that m^* is equal to the mean field. Consider two circles of radii unity and a respectively ($a \le 1$) with their centers separated by m^* on the x-axis as shown in figure (1). The spin lies on the larger circle and the random field on the smaller one. The spin making an angle θ with the x-axis generally cuts the smaller circle at two points. Correspondingly two orientations of the random field, one making an angle α and the other $\alpha + \beta$ with the x-axis produce the same magnetization $m = \cos\theta$. Using the properties of triangles in figure (1), we get $\beta + 2(\alpha - \theta) = \pi$ and $\sin(\pi - \alpha + \theta)/m^* = \sin\theta/a$. These two relations can be combined to give,

$$m^2 = 1 - \left(\frac{a}{m^*}\right)^2 \cos^2\frac{\beta}{2} \tag{9}$$

In the limiting case $\beta = 0$ when the spin vector is tangential to the random field circle, m approaches its minimum value m_{min}. For a given a, the fixed point m^* is an average over various values of m ranging from $m = m_{min}$ (for $\beta = 0$) to $m = 1$ corresponding to $\alpha = 0$ and $\beta = \pi$. Note that the integrand in equation (3) has the same value for $\alpha_i = \pi$ as for $\alpha_i = 0$. Figure (2) shows the fixed points of equation (3) with $J_0 = 1$ and $h = 0$ for increasing and decreasing a starting from $m^* = 1$ at $a = 0$ and changing a in small steps δa. At each step, the fixed point at the preceding step is used as a starting point and a is held fixed during the iterative dynamics till a new fixed point is reached. We see that m^* decreases with increasing a; at $a \approx 0.67$, m^* jumps down from $m^* \approx 0.74$ to $m^* = 0$ and

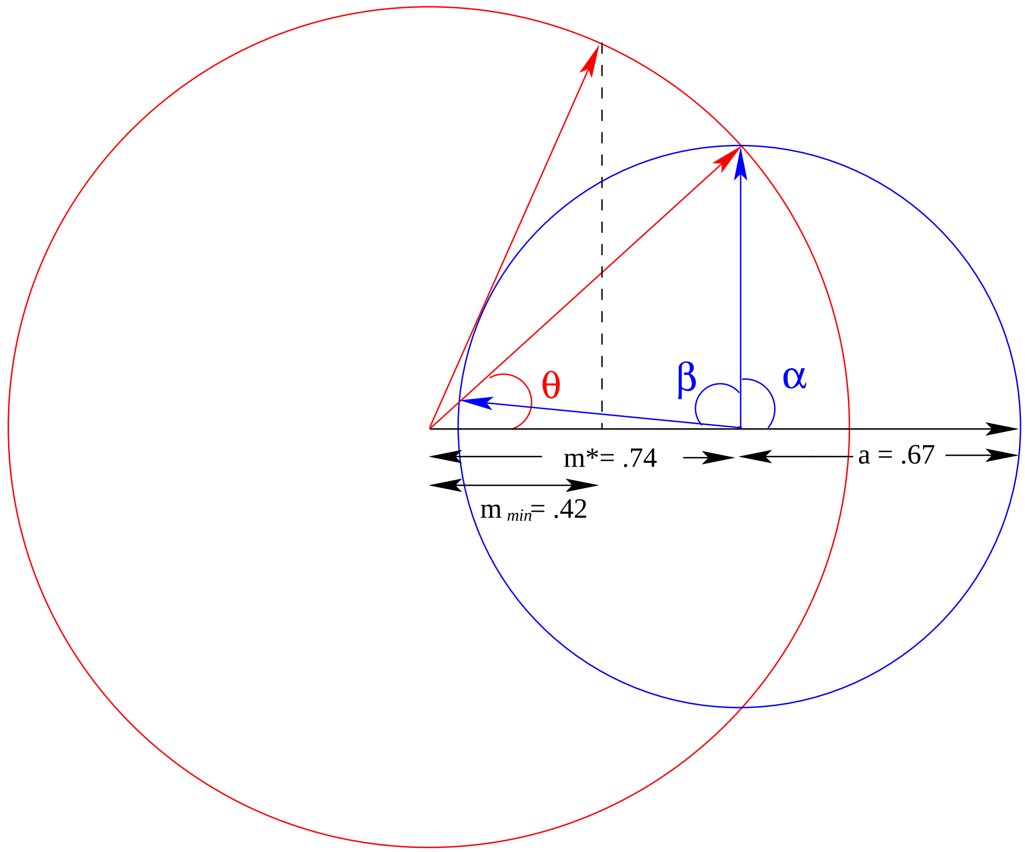

FIG. 1: (Color online) The figure shows a geometrical relationship between the spin, the random field, and the mean field at the threshold of instability in increasing a for $J_0 = 1$ and $h = 0$ (see text).

stays zero thereafter. On decreasing a, m^* stays zero for $a < 0.5$, jumps up to $m^* \approx 0.92$ at $a = 0.5$ and increases along the old trajectory as $a \to 0$. The region $0.5 \le a \le 0.671$ is bistable. It contains two lines of stable fixed points separated by a line of unstable fixed points. The unstable fixed points separate the domains of the stable fixed points. In the bistable region, the system is in the domain of $m^* \neq 0$ in increasing a, and $m^* = 0$ in decreasing a. This is the basic mechanism of hysteresis in our model. The boundaries of the bistable region can be understood geometrically. As a increases from zero the random field circle increases and its center shifts closer to the center of the larger circle. The angle α corresponding to m^* also increases until $\alpha = \pi/2$ at $m^* \approx 0.74$ at $a \approx 0.67$. This is depicted in figure (1). For larger a, $m^* = 0$ and the two circles are concentric in the stable state. The first order jump in the magnetization at $\{h = 0, a \approx 0.67\}$ gives rise to hysteresis loops in cyclic fields as shown in figure (3). Note that $m^* = 0$ at $h = 0$ if $a > 0.67$ but this does not mean the absence of hysteresis but rather wasp-waisted hysteresis loops as explained in reference [8].

If the random field at each site has a random orientation α_i and a Gaussian distributed magnitude a_i with mean zero and standard deviation σ, our numerical work suggests that the wasp-waisted hysteresis loops disappear and the first order jumps in the magnetization gradually diminish with increasing σ and vanish as $\sigma \to \sigma_c$ from below. We find $\sigma_c \approx 0.62$. Figure (4) shows the hysteresis loops for two values of σ, (i) $\sigma < \sigma_c$ and (ii) $\sigma > \sigma_c$. The critical behavior may be obtained analytically. It is again useful to set $J_0 = 1$ and $h = 0$ in equation (5), and then expand it in powers of $m(t)$ in the limit $m(t) \to 0$. We split the range of the integral over a_i in two parts (i) zero to $m(t)$ and (ii) $m(t)$ to ∞ and extract the leading terms from both intervals. Thus,

$$m(t+1) = \frac{1}{2\sigma^2}\left[\int_0^{m_i(t)} e^{-\frac{a_i^2}{2\sigma^2}} a_i da_i \left\{2 - \frac{1}{2}\left(\frac{a_i}{m_i(t)}\right)^2\right\} + \int_{m_i(t)}^{\infty} e^{-\frac{a_i^2}{2\sigma^2}} a_i da_i \left\{\left(\frac{m_i(t)}{a_i}\right) + \frac{1}{8}\left(\frac{m_i(t)}{a_i}\right)^3\right\}\right] \tag{10}$$

After performing the Gaussian integrals and simplifying, we get

$$m(t+1) = \sqrt{\frac{\pi}{8}}\left[\left\{\frac{m(t)}{\sigma}\right\} - \frac{1}{8}\left\{\frac{m(t)}{\sigma}\right\}^3 + \ldots\right] \tag{11}$$

This recursion relation shows that σ has a critical value $\sigma_c = \sqrt{\pi/8} \approx 0.6267$; $m^* = 0$ if $\sigma > \sigma_c$; and $m^* = \sqrt{8\sigma_c}(\sigma_c - \sigma)^{1/2}$ as $\sigma \to \sigma_c$ from below. Thus there is a second order phase transition at $\sigma = \sigma_c$. These predictions are born out by numerical solution of the equation as shown in figure (5).

For randomly oriented fixed-length fields $\{a_i = a\}$, the corresponding equation for the magnetization is,

$$m(t+1) = \frac{1}{2}\left[\left\{\frac{m(t)}{a}\right\} + \frac{1}{8}\left\{\frac{m(t)}{a}\right\}^3 + \ldots\right] \tag{12}$$

As explained in reference [8], the positive sign of the cubic term in the above equation rules out a continuous transition from $m^* = 0$. There is a first order transition in this case. One may ask if the difference in hysteresis for a Delta function distribution $\{a_i = a\}$ and a Gaussian distribution comes from the difference between a sharply localized distribution and an unbounded distribution. In order to examine this question we first relax the equality $a_i = a$ and allow a_i to be uniformly distributed in a narrow range $[a - \epsilon$ to $a + \epsilon]$. We find that the hysteresis for a narrow rectangular distribution around $a_i = a$ is qualitatively the same as that for $a_i = a$ in the limit $\epsilon \to 0$. In the same vein, we also consider the distribution $a_i = \frac{1}{\Delta}$ if $0 \le a_i \le \Delta$, and $a_i = 0$ otherwise. For this distribution, equation (7) takes the following form in the limit $m(t) \to 0$,

$$m(t+1) = \frac{m(t)}{\Delta}\left[1 - \frac{1}{8}\left\{\frac{m(t)}{\Delta}\right\}^2 + \ldots\right] \tag{13}$$

The above recursion relation has a critical point at $\Delta_c = 1$; $m^* = 0$ if $\Delta \ge \Delta_c$, and $m^* = \sqrt{8\Delta^2(\Delta_c - \Delta)}$ as Δ approaches Δ_c from below. Figure (6) shows $m^*(h = 0)$ as Δ increases from zero to $\Delta > \Delta_c$. Thus the critical behavior of the model with a uniform bounded distribution of $\{a_i\}$ is the same as for a Gaussian distribution.

IV. HYSTERESIS IN HEISENBERG MODEL

We now examine hysteresis in the Heisenberg model with each of the three distributions considered above. For random field vectors of a fixed length $|\vec{a_i}| = a$ and $A(t) = J_0 m(t) + h$, equation (4) simplifies to the following:

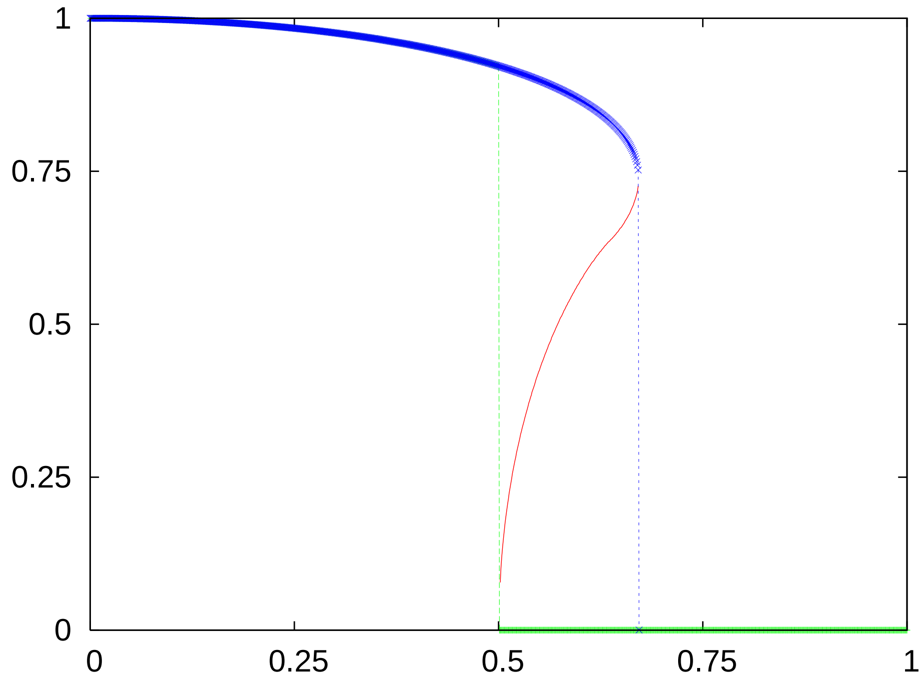

FIG. 2: (Color online) Magnetization curves in increasing and decreasing a at $h = 0$. The thick dark (blue) line shows the magnetization in increasing a. Starting with $m = 1$ at $a = 0$, it decreases with increasing a, drops to zero at $a \approx 0.67$, and stays zero for larger values of a. For decreasing a, it remains zero for $a < 0.5$ along the thick gray (green) line, jumps up on the thick dark (blue) curve at $a = 0.5$ and follows it till the end. The thin dark curve (red) shows a line of unstable fixed points in the bistable region $0.5 \leq a \leq 0.671$. On decreasing a, the starting magnetization is always below the line of unstable fixed points, and therefore in the domain of the $m^* = 0$ fixed point.

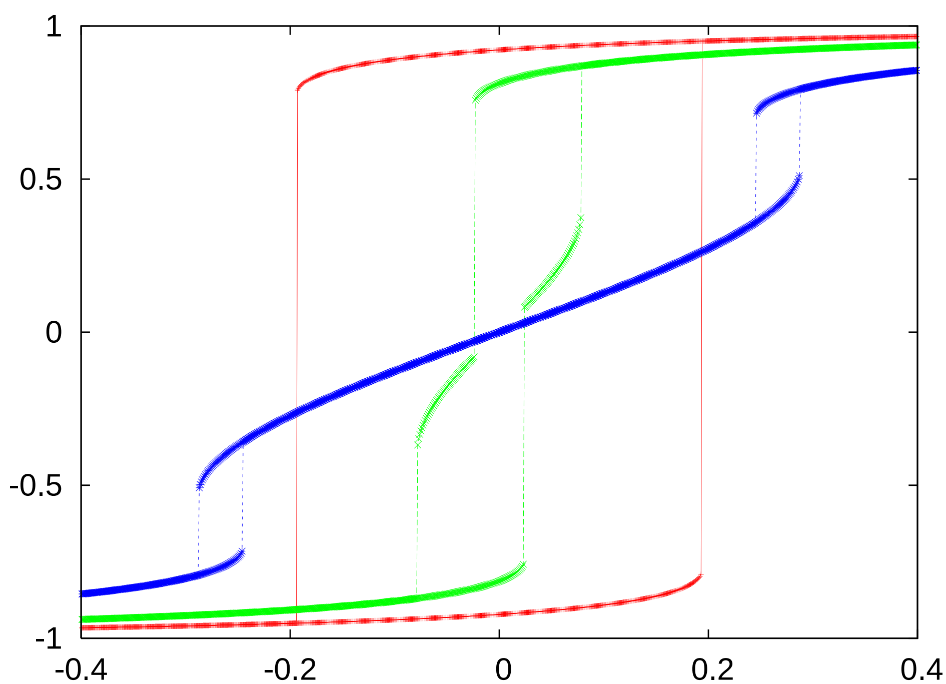

FIG. 3: (Color online) Hysteresis loops for the XY model with randomly oriented fields of length a=0.5 (dark/red line), 0.65 (thicker gray/green line), and 0.9 (thicker dark/blue line). At $a = a_c(\approx 0.67)$, $|m(h = 0)|$ jumps from $m \approx 0.74$ to $m = 0$ (see figure 2). Therefore hysteresis for $a = 0.9$ has a wasp-waisted loop with zero hysteresis in small applied fields around $h = 0$.

$$
\begin{aligned}
m(t+1) &= \frac{2A(t)}{3a} \text{ if } |A(t)| \le a \\
&= 1 - \frac{1}{3}\left\{\frac{a}{A(t)}\right\}^2 \text{ if } A(t) > a \\
&= -1 + \frac{1}{3}\left\{\frac{a}{A(t)}\right\}^2 \text{ if } A(t) < -a
\end{aligned} \tag{14}
$$

Setting $J_0 = 1$ and $h = 0$, the fixed point in the small m limit is determined by the recursion relation $m(t+1) = 2m(t)/3a$. Thus $m^* = 0$ is stable fixed point if $a > 2/3$. At $a = 2/3$ any value in the range $-\frac{2}{3} \le m^* \le \frac{2}{3}$ satisfies the fixed point equation at $h = 0$. This is a peculiarity of mean field hysteresis in the Heisenberg model with randomly oriented fields of fixed length a. If $a > \frac{2}{3}$, $|m^*| > \frac{2}{3}$ and approaches unity as $a \to 0$. Therefore as h is cycled between $-\infty$ and ∞, we get hysteresis if $a < 2/3$, no hysteresis if $a > 2/3$, but hysteresis does not vanish at $a = 2/3$ in any familiar fashion of first or second order transition.

If the magnitude of the random field has a Gaussian distribution, the appropriate recursion relation in the small m limit is obtained from equation (6). Setting $J_0 = 1$ and $h = 0$, we get

$$
m(t+1) = \sqrt{\frac{2}{\pi}}\frac{1}{\sigma^3}\left[\int_0^{m_i(t)} e^{-\frac{a_i^2}{2\sigma^2}} a_i^2 da_i \left\{1 - \frac{1}{3}\left(\frac{a_i}{m_i(t)}\right)^2\right\} + \int_{m_i(t)}^{\infty} e^{-\frac{a_i^2}{2\sigma^2}} a_i^2 da_i \left\{\frac{2m_i(t)}{3a_i}\right\}\right] \tag{15}
$$

This simplifies to,

$$
m(t+1) = \sqrt{\frac{8}{9\pi}}\left[\left\{\frac{m(t)}{\sigma}\right\} - \frac{1}{10}\left\{\frac{m(t)}{\sigma}\right\}^3 + \ldots\right] \tag{16}
$$

In contrast to equation (14), the above equation shows a square root singularity in critical hysteresis at $\sigma_c = \sqrt{8/9\pi} \approx 0.5319$; $m^* = 0$ if $\sigma > \sigma_c$; and $m^* = \sqrt{10\sigma_c}(\sigma_c - \sigma)^{1/2}$ as $\sigma \to \sigma_c$ from below. Therefore magnetization curves for $\sigma > \sigma_c$ should be reversible and those for $\sigma < \sigma_c$ should show hysteresis loops. Figure (4) shows the results for $\sigma = 0.5$ and $\sigma = 0.7$ respectively. Figure (5) shows the critical behavior of $m^*(h = 0)$ as $\sigma \to \sigma_c$. The fit with the expression derived above is good over a rather wide region $\sigma \le \sigma_c$. For comparison, we have also shown the result for the Gaussian random field Ising model in figure (5). In the mean field theory, the magnetization vanishes with a square root singularity as $\sigma \to \sigma_c$ irrespective of n ($n = 1, 2, 3$) but σ_c decreases with increasing n. This may be expected because spins with larger number of components have more freedom to disorder. The Gaussian random field Ising model has been investigated extensively analytically as well as numerically. Its predictions are in reasonable agreement with the experimental observations. It is argued [6] that even for XY and Heisenberg models, if the critical hysteresis occurs at $m^* \ne 0$ or $h \ne 0$, it may show critical exponents of the Ising model because the non-zero magnetization or applied field picks a unique direction in the system. It is not clear why the Gaussian distribution should be best suited for comparison with experiments, but there is comparatively little study of other distributions of the random field. A numerical study shows that critical exponents of three dimensional random field Ising model with a Gaussian distribution are significantly different from those of the same model with a bimodal distribution of random fields [12].

In the case of a rectangular distribution of a_i, the equations are easily integrated. We get,

$$
\begin{aligned}
m(t+1) &= \frac{1}{\Delta}m(t) - \frac{1}{5}\left\{\frac{m(t)}{\Delta}\right\}^3 \text{ if } m(t) \le \Delta \\
&= 1 - \frac{1}{5}\left\{\frac{\Delta}{m(t)}\right\}^2 \text{ if } m(t) > \Delta
\end{aligned} \tag{17}
$$

These equations show that similar to the case of XY model with a rectangular distribution of a_i, the Heisenberg model shows a critical point at $\Delta_c = 1$. The fixed point is $m^* = \sqrt{5(\Delta_c - \Delta)\Delta^2}$ if $m^* \le \Delta$ giving a square root singularity similar to that for a Gaussian distribution. If $m^* > \Delta$, m^* is determined by the real stable root of the cubic equation $m^3 - m^2 + \frac{1}{5}\Delta^2 = 0$. Figure (6) shows the fixed point magnetization at $h = 0$ as Δ is increased from zero to $\Delta > \Delta_c$.

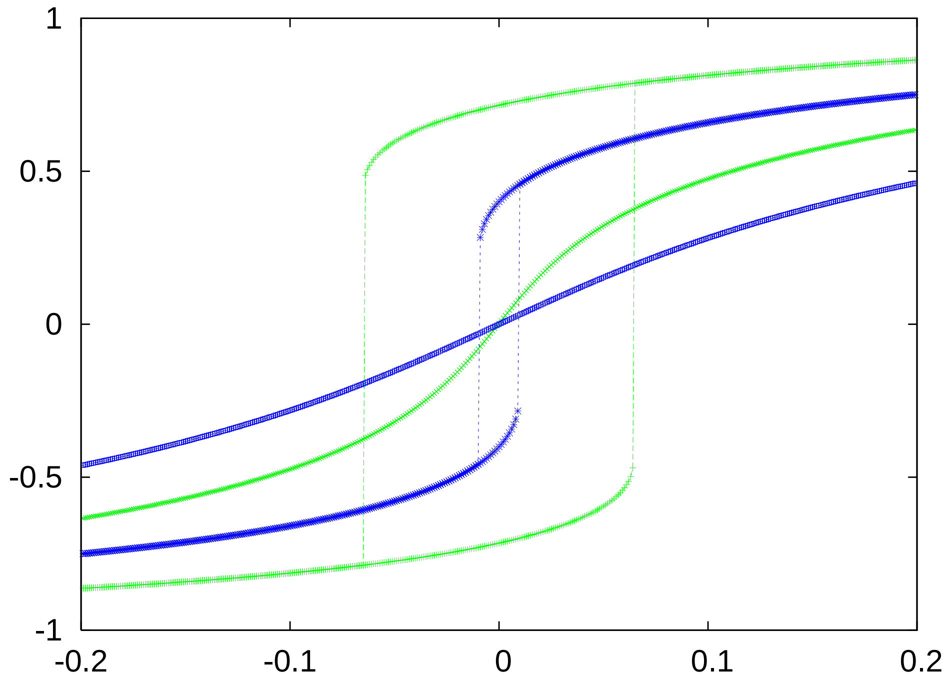

FIG. 4: (Color online) Hysteresis in XY and Heisenberg models with randomly oriented fields of Gaussian magnitude. Results are shown for two values of σ in each case; $\sigma = 0.5$ ($\sigma < \sigma_c$) and $\sigma = 0.7$ ($\sigma > \sigma_c$). The gray (green) curves show the hysteresis loop and the reversible magnetization for the XY model for $\sigma = 0.5$ and $\sigma = 0.7$ respectively. The dark (blue) curves show similar results for the Heisenberg model. The Heisenberg model has a smaller hysteresis loop for $\sigma = 0.5$ and its reversible magnetization for $\sigma = 0.7$ has a smaller slope at $h = 0$.

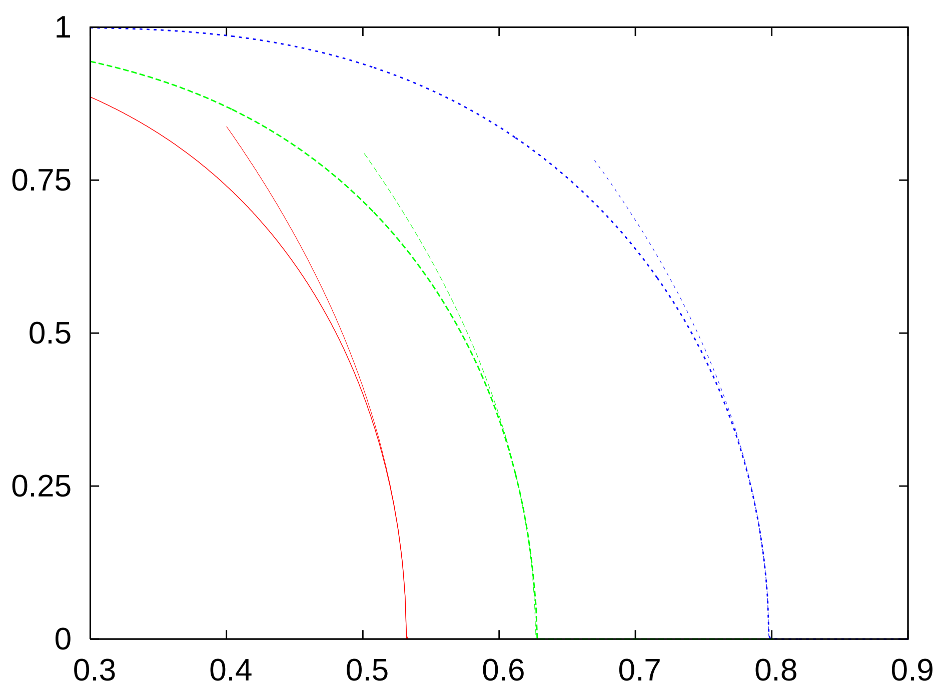

FIG. 5: (Color online) Fixed point magnetization $m^*(h = 0)$ vs σ starting with $m = 1$: Ising model (the curve to the right/blue; $\sigma_c = \sqrt{\pi/2} \approx 0.8$), XY model (the middle curve/green; $\sigma_c = \sqrt{\pi/8} \approx 0.6267$), and Heisenberg model (the curve to the left/red: $\sigma_c = \sqrt{8/9\pi} \approx 0.5319$). The expression $(\sigma_c - \sigma)^{1/2}$ with prefactors $\sqrt{6\sigma_c}$, $\sqrt{8\sigma_c}$, $\sqrt{10\sigma_c}$ (see text) is superimposed on each curve respectively. The square root singularity is seen to fit the data over a rather wide critical region.

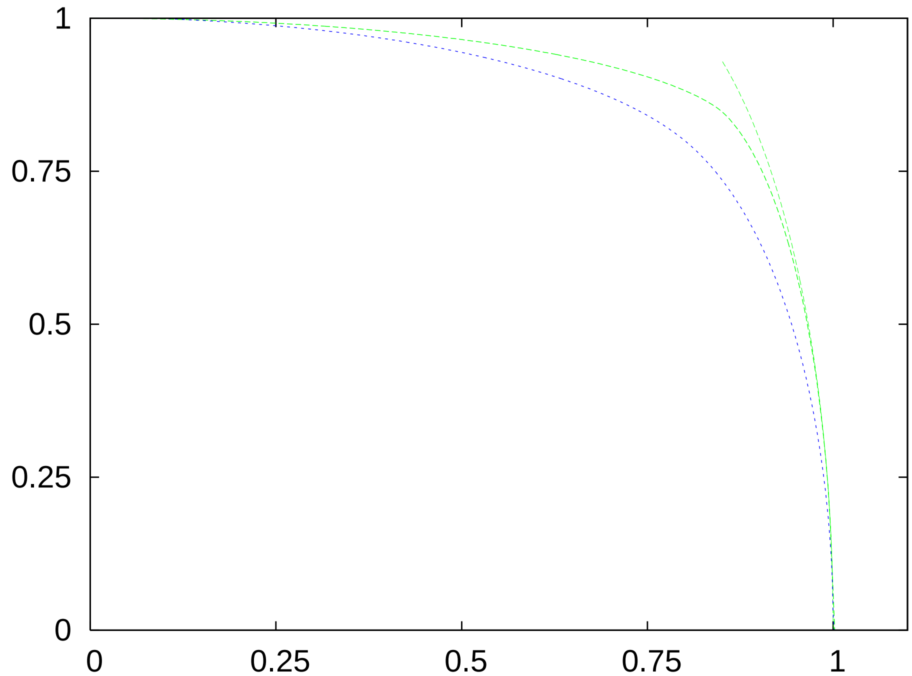

FIG. 6: (Color online) Fixed point magnetization $m^*(h=0)$ vs Δ starting with $m=1$: XY model (the gray/green curve; $\Delta_c = 1$), and Heisenberg model (dark/blue curve: $\Delta_c = 1$). The expression $\sqrt{8}\Delta(\Delta_c - \Delta)^{1/2}$ is superimposed on the XY result to bring out the square root singularity in the critical region. For the Heisenberg model $m^* = \sqrt{5\Delta^2(\Delta_c - \Delta)}$ fits the curve exactly for $m^* \leq \Delta$ (see text).

V. CONCLUDING REMARKS

This work has been motivated by the need to resolve the discrepancy between two exactly solved models of critical hysteresis in n-component spins ($n = 2, 3$) placed in a quenched random field. A calculation [6] based on the RG approach predicts a second order transition with n-independent critical exponents if $d \geq 6$. This calculation uses a Gaussian distribution of the random field. The second calculation [8] does not involve d explicitly but considers the mean field approach of each spin interacting equally with every other spin. It employs randomly oriented unit vector fields and predicts a first order transition and wasp-waisted loops if $n = 2$, and a peculiar transition if $n = 3$. We have shown that if the random fields in the second variant of the model have a Gaussian distribution, the mean field theory reproduces the results of the RG approach. This is reassuring and may have been anticipated. However an explicit verification is valuable because the two models and methods of solution are not identical. The RG approach uses variable length spins with nearest neighbor interactions on a d-dimensional lattice while the spins in the mean field approach are unit vectors. The striking difference in hysteresis between Delta function and Gaussian distribution of the magnitude of random fields is not generic to a localized and an unbounded distribution. We find that a rectangular distribution with a compact support yields the same critical behavior as the Gaussian distribution in the mean field theory. Thus the calculation presented here restores some expectations of the universality of non-equilibrium critical behavior but it also raises other questions that we discuss in the following. It brings out an unexpectedly strong dependence of the shape of hysteresis loops and critical hysteresis on the distribution of the quenched field in the system at the mean field level. These results have bearings on hysteresis in magnetic systems beyond the mean field theory [2, 3, 6–8, 13–17] as well as other adiabatically driven disordered systems [18–26]. We hope the analysis presented here may contribute to a better understanding of hysteresis experiments and also motivate future work to explore what features of quenched disorder are crucial in determining the observed phenomena.

It would be interesting to understand our analytic results on general grounds such as the symmetry of the system and universality classes of critical behavior. Why is it that randomly oriented fixed-length fields produce different critical behavior than randomly oriented fields with Gaussian distributed magnitudes? Why is it that Gaussian distributed magnitudes produce the same critical behavior as uniformly distributed magnitudes? We know that critical hysteresis for Ising spins is different for a uniform bounded distribution of fields than for a Gaussian distribution [5, 9]. An exact solution [5] of the model on a Bethe lattice with coordination number z reveals that there are no jumps in the magnetization nor there is a critical point on the hysteresis loop if the fields have a Gaussian distribution and $z \leq 3$. For a uniform bounded distribution of fields there are jumps in the magnetization even for $z = 3$, and the jumps go to zero *discontinuously* as the width of the distribution is increased. This is due to instabilities in the equations of motion at large quenched fields on the boundary of the distribution. There is also a renormalization group argument [4, 9] that for Ising spins the crucial feature of the probability distribution of random field is its second derivative at zero field. In this formalism the difference in the critical behavior of the models with uniform and Gaussian distributions is attributed to the difference in the second derivative of the probability distribution of the random field at zero field. Although the renormalization group analysis as well as an explicit solution of equations of motion with a given initial condition lead to similar conclusions for critical hysteresis in the case of Ising spins but it does not suggest any simple renormalization group argument why uniform and Gaussian distributions should yield the same critical hysteresis in XY and Heisenberg models. The results presented here are based on an exact solution of the equations of motion in the mean field approximation. The renormalization group tackles the problem rather indirectly. It recasts the equations of motion into a path integral that is a sum over all paths of the exponential of an effective action. The effective action carries extra baggage compared with the equations of motion. This baggage (extra terms and parameters) comes from the use of soft spins and auxiliary fields needed in the exponentiation of delta functions. It is expected that the extra terms are irrelevant in the renormalization group sense and do not influence the critical behavior of the model. However it is not clear to us at this time if the difference between a uniform and Gaussian distribution of quenched fields is irrelevant for spins and fields having more than one component.

An intuitive perspective of results presented in this paper may be obtained by paraphrasing them in terms of energy barriers in the dynamics of n-component spins in quenched fields comprising randomly oriented n-component ($n \geq 2$) vectors of length a. Specifically, consider the dynamics of the system in the absence of an applied field for two different initial conditions (i) all spins pointing along the negative x-axis ($m_0 = -1$), and (ii) all spins pointing along the positive x-axis ($m_0 = 1$). In each case the relaxation dynamics takes the system to a stable fixed point whose magnetization is close to the initial magnetization if $a << J$ where J is the ferromagnetic exchange coupling between spins. The two fixed points are separated by a large gap in their magnetizations and consequently by a large energy barrier between them. As a is gradually increased (keeping J fixed) the gap between the two fixed points decreases. This is understandable because larger a means larger disorder in the system and the fixed point is expected to move farther away from the initial uniform state. In the mean field theory of XY spins, the gap goes to zero *discontinuously* at a critical value of $a = a_c$. If the lengths of the random field vectors $\{a_i\}$ have a uniform distribution in the range $0 \leq a_i \leq \Delta$ or a Gaussian distribution in the range $0 \leq a_i \leq \infty$ with standard deviation σ,

the gap goes to zero *continuously* at a critical value of Δ or σ as the width of the distribution is gradually increased. This too is understandable because the distributed magnitudes, particularly very small magnitudes in the vicinity of zero magnitude would lower the energy barriers for rotation of spins at their sites. The barrier for rotation of a spin depends on the magnitude and direction of the quenched field at its site as well as the orientation of its neighbors. A connected cluster of low a_i sites is likely to have a much lower barrier for collective movement in an avalanche than if all a_i were equal to each other and of the order of J. We may also expect the difference between uniform and Gaussian distributions of $\{a_i\}$ to be less significant for XY spins than for Ising spins that can only flip but not rotate. Similar considerations apply for Heisenberg spins as well where the distributed magnitudes $\{a_i\}$ would push the energy barriers even lower due to an additional degree of freedom for rotation. In a future study we plan to investigate the extent to which the predictions of the mean field theory may apply to hysteresis in random field XY and Heisenberg models with short range interactions on periodic lattices.

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